Information Geometry of Reversible Markov Chains

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Introduction & Preliminaries

Information geometry

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Fisher information metric g and dual affine connections $\nabla^{(e)}$, $\nabla^{(m)}$

$$\mathrm{g}_{ij}(\mu_\theta) := \sum_{x \in \mathcal{X}} \mu_\theta(x)\, \partial_i \log \mu_\theta(x)\, \partial_j \log \mu_\theta(x),$$

$$\Gamma^{(e)}_{ij,k}(\mu_\theta) := \sum_{x \in \mathcal{X}} \partial_i \partial_j \log \mu_\theta(x)\, \partial_k \mu_\theta(x),$$

$$\Gamma^{(m)}_{ij,k}(\mu_\theta) := \sum_{x \in \mathcal{X}} \partial_i \partial_j \mu_\theta(x)\, \partial_k \log \mu_\theta(x).$$

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Wide range of applications

1. Higher-order efficiency analysis of estimator
2. Information decomposition / projection
3. Natural gradient algorithms

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What about Markov models?

Irreducible Markov chains

Notation

$E \quad X^2$ such that (X, E) strongly connected.

Functions and positive functions over E: $F(X, E)$, $F_+(X, E)$.

Irreducible row-stochastic matrices over (X, E): $W(X, E)$.

Often we will take $E = X^2$.

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Discrete-time, time-homogeneous Markov chain

$$\mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = \mu(x_1) \prod_{t=1}^{k-1} P(x_t, x_{t+1}),$$

$$(\mu, P) \in (\mathcal{P}(X), W(X, E)).$$

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Stationary distribution: $\pi P = \pi$.

Edge-measure: $Q(x, x') = \pi(x) P(x, x') = \mathbb{P}_\pi(X_t = x, X_{t+1} = x')$.

Reversible Markov chains

Notation

Time-reversal: $P^{?}(x, x') = p(x')P(x', x)/p(x)$.

Reversible: $p(x)P(x, x') = p(x')P(x', x)$ (detailed balance).

$W_{\text{rev}}(X, E)$ the set of all reversible kernels.

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Numerous interesting properties (e.g.)

Spectrum is real and stable.

Mixing time is governed by spectral gap.

Stationary distribution is easy to compute.

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Historical remarks

Inspiration & origin Schrödinger (1931).

First treatment for MCs Kolmogorov (1936, 1937).

Tribute and historical summary Dobrushin et al. (1988).

Exponential tilting (ET)

ET of distribution
$X \sim m \in \mathcal{P}([m]), f : X \to \mathbb{R}$. Construct exponential family:

$$m_q(x) = m(x)e^{qf(x) - k(q)}, \qquad k(q) = \log \mathrm{E}\, e^{qf(X)} \qquad \text{(CGF)}.$$

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$$\lambda > \mathbb{E}[f], \lim_{k \to \infty} \frac{1}{k} \log \mathbb{P}\left(\frac{1}{k} \sum_{t=1}^{k} f(X_t) > \lambda \right) = -\kappa^*(\lambda), \quad \sup_{\theta \in \mathbb{R}} \{\theta\lambda - \kappa(\theta)\}.$$

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$P \in \mathcal{W}$, $f : \mathcal{X} \to \mathbb{R}$, $\tilde{P}_\theta(x, x') = P(x, x') \exp(\theta f(x'))$, $P_\theta = s(\tilde{P}_\theta)$, with Perron-Frobenius (PF) rescaling (Miller, 1961),

$$s : \mathcal{F}_+(\mathcal{X}, \mathcal{E}) \to \mathcal{W}(\mathcal{X}, \mathcal{E}), \ \tilde{P}(x, x') \mapsto P(x, x') = \frac{\tilde{P}(x, x') v(x')}{\rho v(x)},$$

where ρ, v are the PF root and right PF eigenvector of $\tilde{P}$.

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Information Geometry of Markov Chains

Information Geometry of Markov Chains

1. Large deviations: Miller (1961); Donsker and Varadhan (1975); Gärtner (1977)
2. Information projection: Csiszár et al. (1987)
3. Asymptotic e-families: Ito and Amari (1988); Takeuchi and Barron (1998); Takeuchi and Kawabata (2007); Takeuchi and Nagaoka (2017)
4. One-parameters exponential families: Nakagawa and Kanaya (1993)
5. Dually flat structure: Nagaoka (2005)

Information Geometry of Markov Chains (Nagaoka, 2005)

Distributions

$P(X)$

D

KL divergence

$$D(m_\theta \| m_{\theta'}) = E_{m_\theta} \log \frac{m_\theta(x)}{m_\theta(x')}$$

Information Geometry of Markov Chains (Nagaoka, 2005)

Distributions $\mathcal{P}(\mathcal{X})$

Markov chains $\mathcal{W}(\mathcal{X}, \mathcal{E})$

$$X_1, X_2, \dots, X_t \quad P$$

$$D \longrightarrow D$$

KL divergence

KL divergence rate

$$\lim_{t \to \infty} \frac{1}{t} D\left(X_1, \dots, X_t \quad P_\theta \| X'_1, \dots, X'_t \quad P_{\theta'}\right) = \mathbb{E}_{(X,X') \quad Q_\theta} \log \frac{P_\theta(X,X')}{P_{\theta'}(X,X')}$$

$$\triangleq D\left(P_\theta \| P_{\theta'}\right)$$

Information Geometry of Markov Chains (Nagaoka, 2005)

Distributions	Markov chains
$P(X)$	$W(X, E)$

limit ¥

$r^{(e)}, r^{(m)}$ ⟶ $r^{(e)}, r^{(m)}$

g ⟶ g

D ⟶ D

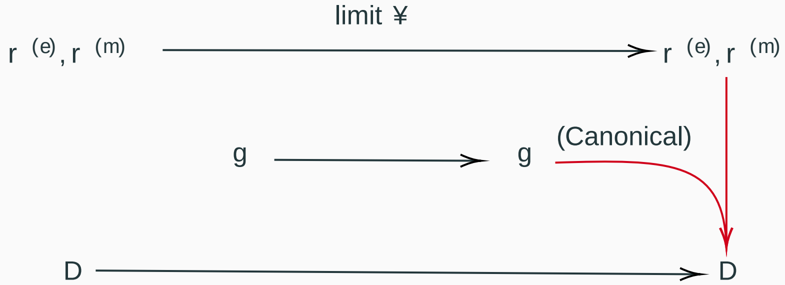
Information Geometry of Markov Chains (Nagaoka, 2005)

Distributions

$P(X)$

Markov chains

$W(X, E)$



View $\mathcal{W}(\mathcal{X}, \mathcal{E})$ as a smooth manifold.

Fisher information metric $\mathfrak{g}$

$$g_{ij}(P_\theta) \triangleq \sum_{(x,x')\in\mathcal{E}} Q_\theta(x,x')\,\partial_i \log P_\theta(x,x')\,\partial_j \log P_\theta(x,x').$$

Dual affine connections $\nabla^{(e)}, \nabla^{(m)}$

$$\Gamma^{(e)}_{ij,k}(P_\theta) \triangleq \sum_{(x,x')\in\mathcal{E}} \partial_i\partial_j \log P_\theta(x,x')\,\partial_k Q_\theta(x,x'),$$

$$\Gamma^{(m)}_{ij,k}(P_\theta) \triangleq \sum_{(x,x')\in\mathcal{E}} \partial_i\partial_j Q_\theta(x,x')\,\partial_k \log P_\theta(x,x').$$

Exponential families of transition kernels (Nagaoka, 2005)

Let $\Theta \subseteq \mathbb{R}^d$, open connected parameter space.

$$\mathcal{V}_e = \left\{ P_\theta : \theta = (\theta^1, \ldots, \theta^d) \in \Theta \right\} \subset \mathcal{W}$$

is an e-family with natural parameter θ, whenever there exist functions $K, R_\theta, \psi_\theta, g_1, \ldots, g_d$ such that

$$\log P_\theta(x, x') = K(x, x') + \sum_{i=1}^{d} \theta^i g_i(x, x') + \underbrace{R_\theta(x') - R_\theta(x) - \psi_\theta}_{\text{rescaling terms}} .$$

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Example 1 (Nagaoka, 2005)

$\mathcal{W}(\mathcal{X}, \mathcal{E})$ forms an e-family of dimension $|\mathcal{E}| - |\mathcal{X}|$. For $\mathcal{E} = \mathcal{X}^2$,

$$\log P(x, x') = \sum_{i=1}^{|\mathcal{X}|} \sum_{\substack{j=1 \\ j \neq x_\star}}^{|\mathcal{X}|} \overbrace{\log \frac{P(i,j)P(j,x_\star)}{P(i,x_\star)P(x_\star,x_\star)}}^{\theta^{ij}} \overbrace{\delta_i(x)\delta_j(x')}^{g_{ij}(x,x')}$$

$$+ \log P(x, x_\star) - \log P(x', x_\star) + \log P(x_\star, x_\star).$$

Mixture families (Nagaoka, 2005)

We say that $\mathcal{V}_m$ is a mixture family when there exists functions $C, F_1, \ldots, F_d$, such that $C, C + F_1, \ldots, C + F_d$ are affinely independent,

$$\sum_{x,x' \in \mathcal{X}} C(x, x') = 1, \qquad \sum_{x,x' \in \mathcal{X}} F_i(x, x') = 0, \forall i \in [d],$$

and

$$\mathcal{V}_m = \left\{ P_\xi \in \Omega : Q_\xi = C + \sum_{i=1}^{d} \xi^i F_i, \xi \in \Xi \right\}$$

where $\Xi = \left\{ \xi \in \mathbb{R}^d : Q_\xi(x, x') > 0, \forall (x, x') \in \mathcal{X}^2 \right\}$, and Q_ξ is the edge measure that pertains to P_ξ.

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$$V_m = \left\{ P_x \in \Omega : Q_x = C + \sum_{i=1}^{d} x^i F_i, x \in X \right\}$$

where $X = \left\{ x \in \mathbb{R}^d : Q_x(x, x') > 0, \forall (x, x') \in X^2 \right\}$, and Q_x is the edge measure that pertains to P_x.

Example 2
W_{bis} (doubly stochastic) forms an m-family (Hayashi and Watanabe, 2016).

Geometric approach has recently lead to finite sample analysis for:

1. Parameter estimation problem in Markov chains in HMMs Hayashi and Watanabe (2016); Hayashi (2022).

Applications { Inference in Markov chains

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Extension of known conceptsˆ but also new structures emerge!

Time-reversal families

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Let V W .

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$$V_m \text{ is an m-family } =) \; V_m^{?} \text{ is an m-family.}$$

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What about families of fixed points?

Reversible e-families

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Unbiased lazy random walks on the d-cycle form a reversible e-family.

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Problem: How to determine whether a given e-family is reversible ?

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Theorem (Kolmogorov's criterion (Kolmogorov, 1936))

Let P irreducible, and Γ the set of finite directed closed-paths, i.e. $\gamma \in \Gamma$, when $\gamma : [n] \to \mathcal{E}$, $\gamma(t) = (x_t, x_{t+1})$ with $x_{n+1} = x_1$. Write $\gamma^\star$ the time-reversed path. P is reversible if and only if for all $\gamma \in \Gamma$,

$$\prod_{t=1}^{|\gamma|} P(\gamma(t)) = \prod_{t=1}^{|\gamma|} P(\gamma^\star(t)).$$

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$$\prod_{t=1}^{|\gamma|} P(\gamma(t)) = \prod_{t=1}^{|\gamma|} P(\gamma^\star(t)).$$

Log-reversible function
A function $h \in \mathcal{F}$ is log-reversible whenever it satisfies that for all $\gamma \in \Gamma$,

$$\sum_{t=1}^{|\gamma|} h(\gamma(t)) = \sum_{t=1}^{|\gamma|} h(\gamma^\star(t)).$$

Characterization of reversible e-families

Theorem 2

Let V be e-family of kernels generated by K and $(g_i)_{i \in [d]}$. The two statements are equivalent:

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(ii) $\mathcal{V}$ is such that K and $g_i, \forall i \in [d]$ are all log-reversible functions (and $\mathcal{E} = \mathcal{E}^{\star}$ for more general irreducible kernels).

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Naively, the number of distinct equations that must be checked would be large

$$(d+1) \sum_{k=3}^{|\mathcal{X}|} \binom{|\mathcal{X}|}{k} \frac{(k-1)!}{2}.$$

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Lemma 2 (Polynomial time procedure for reversibility checking)

Let $h \in F_+$ (more generally irreducible). $\log[h]$ is log-reversible if and only if $P_h \circ h^\top$ is a symmetric matrix, with $P_h = v_h^\top u_h$ the PF projection of h, where u_h and v_h are respectively the left and right PF eigenvectors of h, normalized such that $u_h v_h^\top = 1$.

Lemma 4 (Vector space of log-reversible functions)

$$\mathcal{F}_{\text{rev}} \triangleq \{ h \in \mathcal{F} : h \text{ is log-reversible} \},$$
$$\mathcal{N} \triangleq \{ h \in \mathcal{F} : \exists (c, f),\, h(x, x') = f(x') - f(x) + c \}.$$

Affine structures

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Quotient space: $\mathcal{G}_{rev} := \mathcal{F}_{rev} / \mathcal{N}$

$\mathcal{D}(\mathcal{G}_{rev}) = \mathcal{W}_{rev}$, with $\mathcal{D}(f) := s(\exp[f])$ diffeomorphism.

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Theorem 3 & 4

Let $T = \{x \to x'\} \setminus \{(m, 1)\}$, $|T| = |X|(|X| + 1)/2 - 1$.

Lemma 4 (Vector space of log-reversible functions)

$$\mathcal{F}_{rev} \coloneqq \{h \in \mathcal{F} : h \text{ is log-reversible}\},$$
$$\mathcal{N} \coloneqq \{h \in \mathcal{F} : \exists (c, f),\ h(x, x') = f(x') - f(x) + c\}.$$

$$\mathcal{N} \subset \mathcal{F}_{rev} \subset \mathcal{F} \quad \text{(vector space inclusions)}.$$

Quotient space: $\mathcal{G}_{rev} \coloneqq \mathcal{F}_{rev} / \mathcal{N}$

$$D(\mathcal{G}_{rev}) = \mathcal{W}_{rev}, \text{ with } D(f) \coloneqq s(\exp[f]) \text{ diffeomorphism}.$$

Theorem 3 & 4
Let $\mathcal{T} = \{x \leq x'\} \setminus \{(m, 1)\}$, $|\mathcal{T}| = |\mathcal{X}|(|\mathcal{X}| + 1)/2 - 1$.

(i) $\mathcal{G}_{rev}$ is a $|\mathcal{T}|$-dimensional vector space.

Affine structures

Lemma 4 (Vector space of log-reversible functions)

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Quotient space: $\mathcal{G}_{\text{rev}} \triangleq \mathcal{F}_{\text{rev}} / \mathcal{N}$

$$\Delta(\mathcal{G}_{\text{rev}}) = \mathcal{W}_{\text{rev}}, \text{ with } \Delta(f) \triangleq \mathfrak{s}(\exp[f]) \text{ diffeomorphism}.$$

Theorem 3 & 4

Let $\mathcal{T} = \{x \to x'\} \setminus \{(m, 1)\}$, $|\mathcal{T}| = |\mathcal{X}|(|\mathcal{X}| + 1)/2 - 1$.

(i) $\mathcal{G}_{\text{rev}}$ is a $|\mathcal{T}|$-dimensional vector space.

(ii) $\mathcal{W}_{\text{rev}}$ is an e-family, and is an m-family.

Affine structures

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Theorem 3 & 4

Let $\mathcal{T} = \{x \to x'\} \setminus \{(m, 1)\}$, $|\mathcal{T}| = |\mathcal{X}|(|\mathcal{X}| + 1)/2 - 1$.

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Affine structures

Lemma 4 (Vector space of log-reversible functions)

$$F_{rev} \triangleq \{ h \in F : h \text{ is log-reversible} \},$$
$$N \triangleq \{ h \in F : \exists (c, f), h(x, x') = f(x') - f(x) + c \}.$$

$$N \subset F_{rev} \subset F \quad \text{(vector space inclusions)}.$$

Quotient space: $G_{rev} \triangleq F_{rev} / N$

$D(G_{rev}) = W_{rev}$, with $D(f) \triangleq s(\exp[f])$ diffeomorphism.

Theorem 3 & 4

Let $T = \{x \leq x'\} \setminus \{(m, 1)\}$, $|T| = |X|(|X| + 1)/2 - 1$.

(i) G_{rev} is a $|T|$-dimensional vector space.

(ii) W_{rev} is an e-family, and is an m-family.

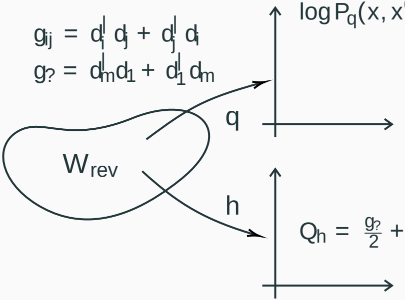
(iii) $\dim W_{rev} = |T|$.

(iv) Basis: $\left\{ g_{ij} \triangleq d_i^\top d_j + d_j^\top d_i : (i, j) \in T \right\}$.

Parametrization of $\mathcal{W}_{rev}$ (Theorem 5)

$$g_{ij} = \delta_i^{\intercal}\delta_j + \delta_j^{\intercal}\delta_i$$

$$g_{?} = \delta_m^{\intercal}\delta_1 + \delta_1^{\intercal}\delta_m$$



$$\log P_\theta(x,x') = \sum_{(i,j)\in T} \overbrace{\frac{1}{2(1+\delta_i(j))}\log\frac{P(i,j)P(j,i)}{P(m,1)P(1,m)}}^{=\theta^{ij}} g_{ij}(x,x') + \frac{1}{2}\log p(x') \quad \frac{1}{2}\log p(x) + \frac{1}{2}\log P(m,1)P(1,m)$$

$$Q_\eta = \frac{g_{?}}{2} + \sum_{(i,j)\in T}(g_{ij} \quad g_{?})\frac{Q(i,j)}{1+\delta_i(j)}$$

$$\theta^i(\eta) = \sum_{x,x'\in E} \frac{Q_\eta(x,x')}{\partial\eta_i} \log P_\eta(x,x')$$

$$\eta_i(\theta) = \sum_{x,x'\in E} Q_\theta(x,x') g_i(x,x')$$

Reversible e/m-projections

Closest reversible chain

Reversiblization
Given an irreducible (non-reversible) P, we can construct

1. $(P + P^{?})/2$, the additive reversiblization of P.
2. $PP^{?}$, the multiplicative reversiblization of P (Fill, 1991).
3. $S_p(P)$, the reversible dilation of P (Wolfer and Kontorovich, 2022).

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Properties of reversible chains are powerful but bri,le

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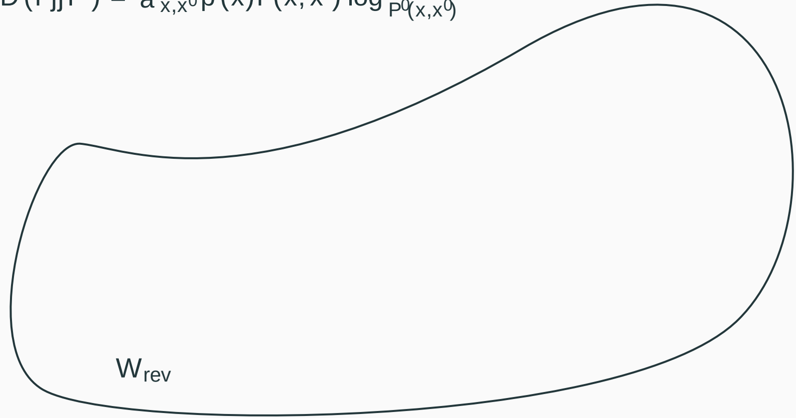
Related work
Analyzed by Nielsen and Weber (2015) for matrix norms induced by scalar products (unique, obtained by solving a convex minimization problem).

Reversible e-projection / m-projection (Theorem 7)

$$D(P \| P') = \sum_{x,x'} \pi(x) P(x,x') \log \frac{P(x,x')}{P'(x,x')}$$



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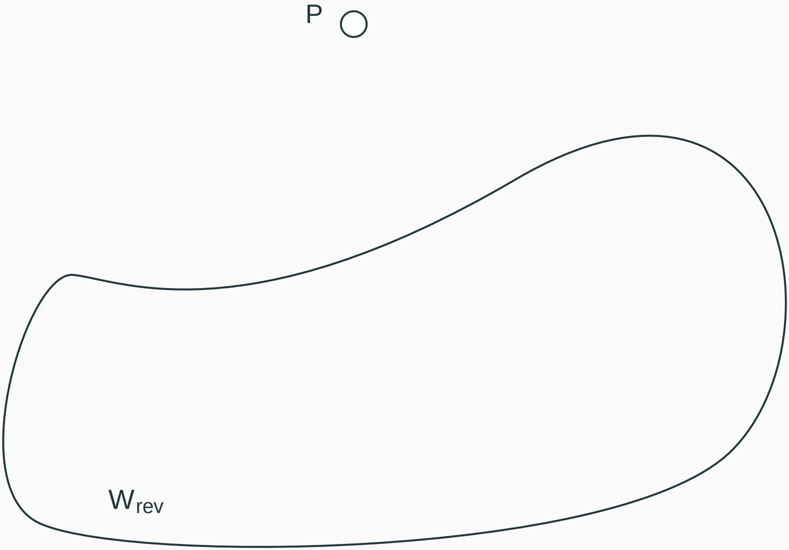
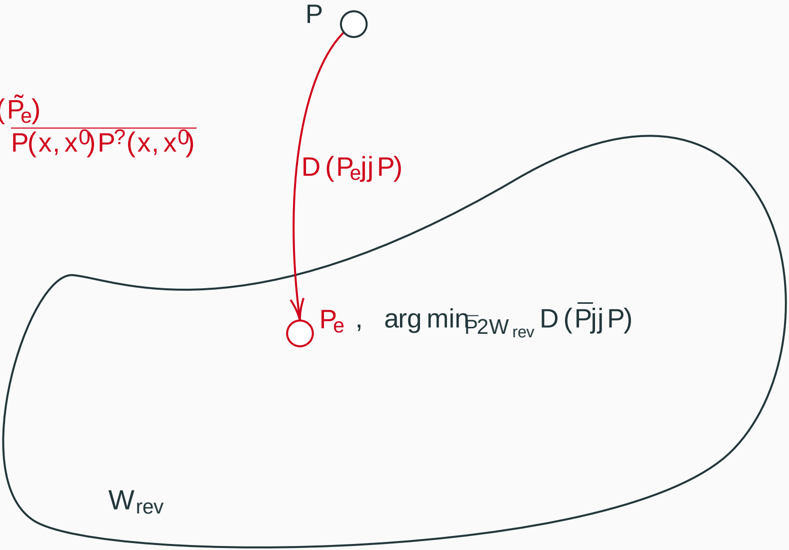


Reversible e-projection / m-projection (Theorem 7)

$$P_e = s(\tilde{P}_e)$$

$$\tilde{P}_e = \sqrt{P(x,x')P^{\star}(x,x')}$$



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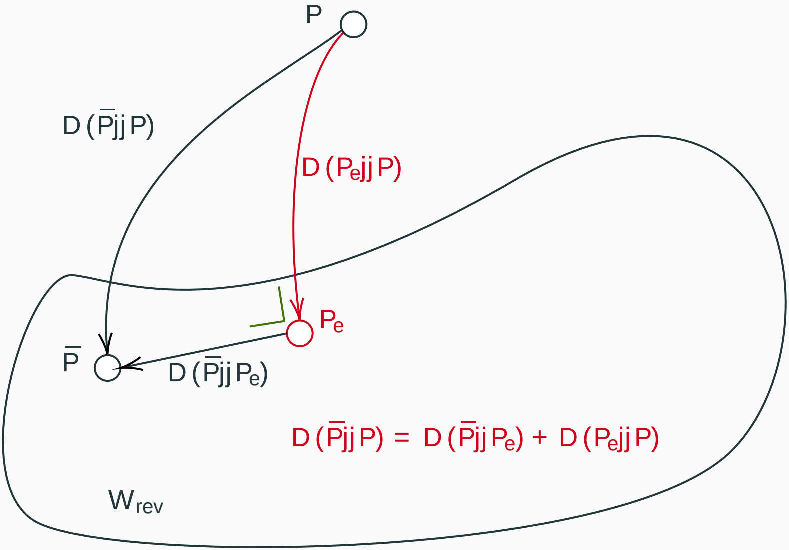
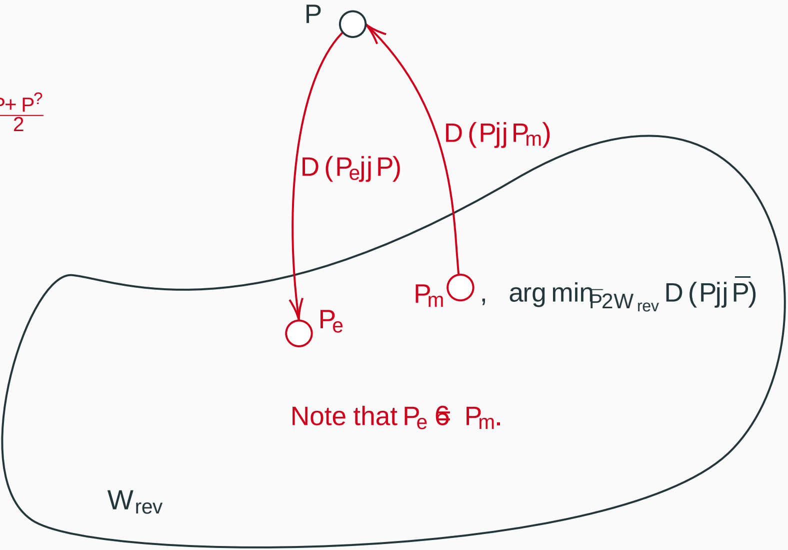


Reversible e-projection / m-projection (Theorem 7)

$$P_m = \frac{P + P^{\star}}{2}$$



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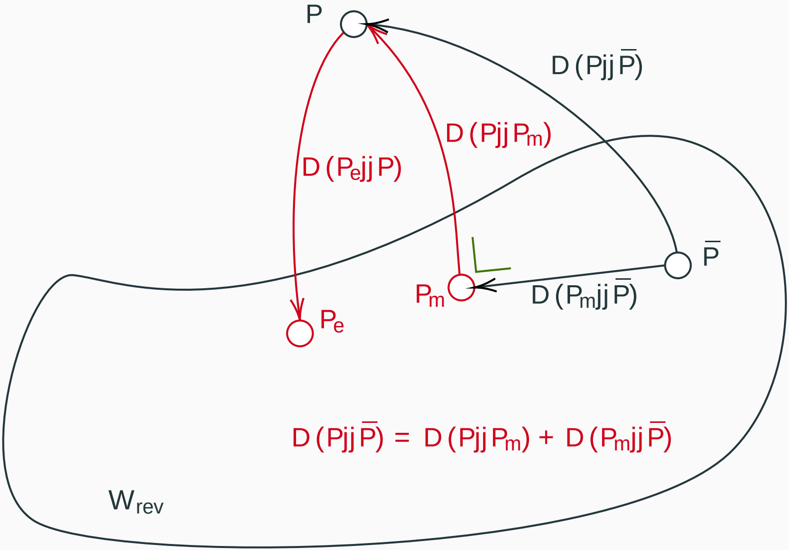

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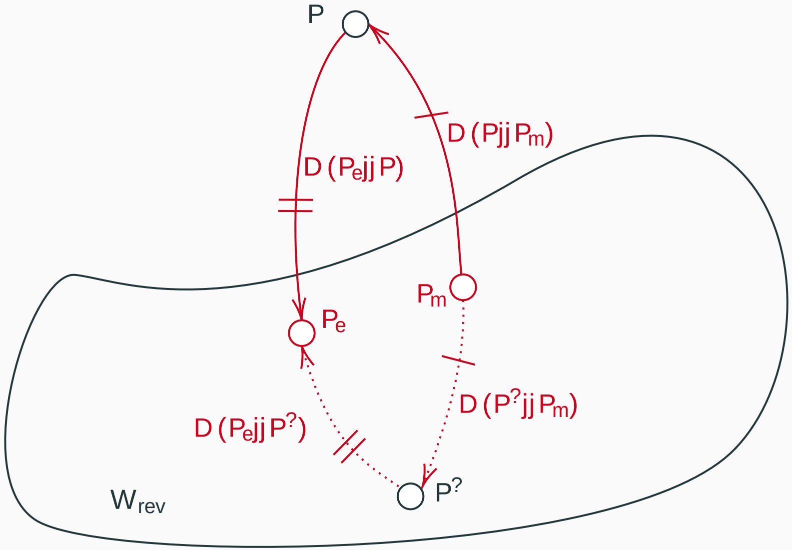

Usual families of kernels

Usual families of kernels

$\mathcal{W}_{bis}$: bi-stochastic kernels ($P^\top \in \mathcal{W}$)
$\mathcal{W}_{sym}$: symmetric kernels ($P = P^\top$)
$\mathcal{W}_{iid}$: memoryless kernels ($P = \mathbf{1}^\top p$)

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Mfd.	m-family	e-family	Dimension
$\mathcal{W}$	✓	✓	$\lvert\mathcal{X}\rvert(\lvert\mathcal{X}\rvert - 1)$
$\mathcal{W}_{rev}$	✓	✓	$\lvert\mathcal{X}\rvert(\lvert\mathcal{X}\rvert + 1)/2 - 1$
$\mathcal{W}_{bis}$	✓	✗	$(\lvert\mathcal{X}\rvert - 1)^2$
$\mathcal{W}_{sym}$	✓	✗	$\lvert\mathcal{X}\rvert(\lvert\mathcal{X}\rvert - 1)/2$
$\mathcal{W}_{iid}$	✗	✓	$\lvert\mathcal{X}\rvert - 1$

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$\mathcal{W}_{bis}$	✓	✗	$(\lvert\mathcal{X}\rvert - 1)^2$
$\mathcal{W}_{sym}$	✓	✗	$\lvert\mathcal{X}\rvert(\lvert\mathcal{X}\rvert - 1)/2$
$\mathcal{W}_{iid}$	✗	✓	$\lvert\mathcal{X}\rvert - 1$

Hierarchies

$$\mathcal{W}_{iid} \overset{\text{e-family}}{\subsetneq} \mathcal{W}_{rev} \overset{\text{e-family}}{\subsetneq} \mathcal{W},$$

$$\mathcal{W}_{sym} \overset{\text{m-family}}{\subsetneq} \mathcal{W}_{rev} \overset{\text{m-family}}{\subsetneq} \mathcal{W},$$

$$\mathcal{W}_{sym} \overset{\text{m-family}}{\subsetneq} \mathcal{W}_{bis} \overset{\text{m-family}}{\subsetneq} \mathcal{W}.$$

Generation of reversible kernels

Generation of the reversible family from W_{iid}

Definition 4 (Mixture hull)
Let $V \quad W$.

$$\text{m-hull}(V) = \left(P : Q \in Q, Q = \sum_{i=1}^{k} a_i Q_i, k \in N, a_1, \ldots, a_k \in R, P_1, \ldots, P_k \in V \right),$$

where Q (resp. Q_i) pertains to P (resp. P_i).

Generation of the reversible family from W_{iid}

Definition 4 (Mixture hull)

Let $V \subseteq W$.

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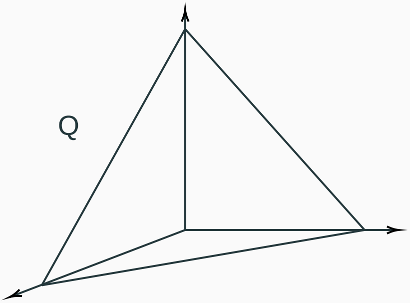

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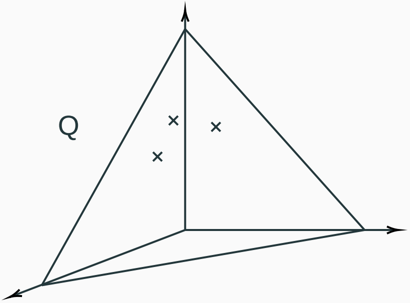

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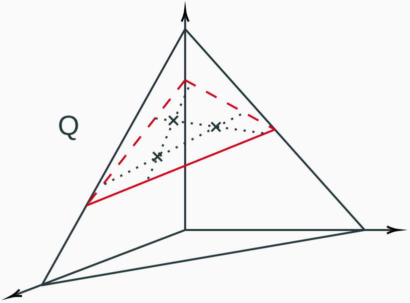

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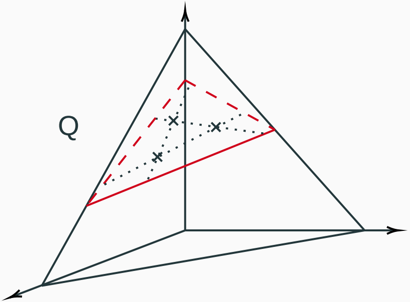

Theorem 10

$$\text{m-hull}(W_{iid}) = W_{rev}.$$

Definition 5 (Exponential hull)

Let $V \subseteq \Omega$.

$$\text{e-hull}(V) = \left\{ \sigma(\tilde{P}) : \log[\tilde{P}] = \sum_{i=1}^{k} \alpha_i \log[P_i], \right.$$

$$\left. k \in \mathbb{N}, \alpha_1, \ldots, \alpha_k \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1, P_1, \ldots P_k \in V \right\}.$$

Generation of the reversible family from W_{sym}

Definition 5 (Exponential hull)
Let $V \subseteq W$.

$$\text{e-hull}(V) = \Big\{ s(P) : \log[P] = \sum_{i=1}^{k} a_i \log[P_i], \; k \in \mathbb{N}, a_1, \ldots, a_k \in \mathbb{R}, \sum_{i=1}^{k} a_i = 1, P_1, \ldots P_k \in V \Big\}.$$

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e-geodesic $g^{(e)}_{P,P'}(t), t \in (-\infty, +\infty)$

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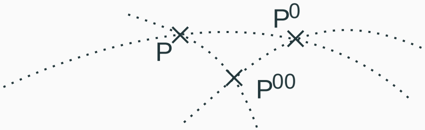

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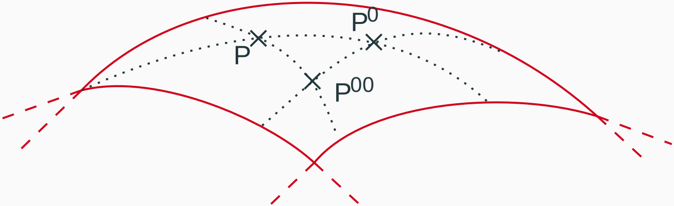

Generation of the reversible family from $\mathcal{W}_{sym}$

Definition 5 (Exponential hull)
Let $\mathcal{V} \subseteq \mathcal{W}$.

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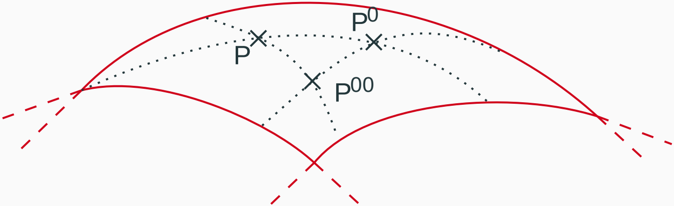
$$\left. k \in \mathbb{N}, \alpha_1, \dots, \alpha_k \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1, P_1, \dots P_k \in \mathcal{V} \right\}.$$



Theorem 9

$$|\mathcal{X}| \geq 3 \implies \text{e-hull}(\mathcal{W}_{sym}) = \mathcal{W}_{rev}.$$

Reversible edge measures

Reversible family of edge measures

Edge-measures

P reversible $\implies Q \iff \operatorname{diag}(\pi) P$ is symmetric.

$$Q = \left(Q : \sum_{x' \in X} Q(x, x') = \sum_{x' \in X} Q(x', x) \right),$$

$$Q_{rev} = \{ Q \in Q : Q^{\star} = Q \}.$$

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$$\mathcal{Q}_{\mathrm{rev}} = \{ Q \in \mathcal{Q} : Q^{\star} = Q \}.$$

Lemma 6

$$\mathcal{Q}_{\mathrm{rev}} \cong \mathcal{P} \left(\frac{|\mathcal{X}| (|\mathcal{X}| + 1)}{2} \right),$$

where $\cong$ denotes Markov equivalence.

Reversible family of edge measures

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$$\mathcal{Q} = \left\{ Q : \sum_{x' \in \mathcal{X}} Q(x, x') = \sum_{x' \in \mathcal{X}} Q(x', x) \right\},$$

$$\mathcal{Q}_{rev} = \{ Q \in \mathcal{Q} : Q^{\star} = Q \}.$$

Lemma 6

$$\mathcal{Q}_{rev} = \mathcal{P} \quad \frac{|\mathcal{X}|(|\mathcal{X}|+1)}{2},$$

where $=$ denotes Markov equivalence.

Theorem 8

(i) $\mathcal{Q}_{rev}$ is e-family and m-family of $\mathcal{P}(\mathcal{X}^2)$.

Edge-measures

P reversible $\Rightarrow$ $Q \Leftrightarrow \mathrm{diag}(\pi)P$ is symmetric.

$$Q = \left(Q : \sum_{x' \in X} Q(x, x') = \sum_{x' \in X} Q(x', x) \right),$$

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Lemma 6

$$Q_{rev} = P \frac{|X|(|X|+1)}{2},$$

where $=$ denotes Markov equivalence.

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(i) Q_{rev} is e-family and m-family of $P(X^2)$.

(ii) $\dim Q_{rev} = |X|(|X|+1)/2 \quad 1$.

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(i) Q_{rev} is e-family and m-family of $P(X^2)$.

(ii) $\dim Q_{rev} = |X|(|X|+1)/2 \quad 1$.

(iii) Q is not e-family in $P(X^2)$ (except when $|X| = 2$).

Summary

(*i*) Property of being an e/m-family is invariant under time-reversal operation.

(*ii*) We introduce reversible e-families, and provide a characterization (efficiently verifiable).

(*iii*) Perhaps surprisingly, W_{rev} forms an e/m-family. We further construct a basis, and derive an explicit parametrization.

(*iv*) We show that e/m-projections, possess closed-form expressions that verify Pythagorean identities, and are always equidistant from an irreducible Markov kernel and its time-reversal (bisection property).

(*v*) We characterize the reversible family as both the minimal exponential family that comprises symmetric kernels, and the smallest mixture family that contains memoryless Markov kernels.

(*vi*) We show that, unlike the set of all irreducible edge measures, the restriction to reversible edge measures forms an e-family (in distributions over pairs).

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Thank you for listening!

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